

TAGGED MAPPING CLASS GROUPS I: AUSLANDER-REITEN TRANSLATION

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ABSTRACT. We give a geometric realization, the tagged rotation, of the AR-translation on Fomin-Shapiro-Thurston's marked surface \mathbf{S} , which generalizes Brüstle-Zhang's result for the puncture free case. As an application, we show that the intersection of the shifts in the 3-Calabi-Yau derived category $\mathcal{D}(\Gamma_{\mathbf{S}})$ associated to the surface and the corresponding Seidel-Thomas braid group of $\mathcal{D}(\Gamma_{\mathbf{S}})$ is empty, unless \mathbf{S} is a polygon with at most one puncture (i.e. of type A or D).

Key words: Mapping cluster group, Auslander-Reiten translation, triangulated surface, cluster theory, braid group

1. INTRODUCTION

Fomin, Shapiro and Thurston studied in [9] the cluster combinatorics of a marked surface, that is, an oriented surface with a finite set of marked points \mathbf{M} on the boundary and a finite set of punctures \mathbf{P} inside \mathbf{S} . In order to allow for flips at all arcs, which corresponds to mutations of cluster variables, they introduced tagged triangulations which are (ideal) triangulations equipped with a tagging at each puncture. The thus obtained exchange graph $\text{EG}^{\times}(\mathbf{S})$ of tagged triangulations, encoding flips of tagged triangulations, is isomorphic to the cluster exchange graph of the cluster algebra defined by the marked surface \mathbf{S} .

Automorphisms of exchange graphs give rise to the group of cluster algebra automorphisms, studied in [2]. Consider the marked mapping class group $\text{MCG}_{\bullet}(\mathbf{S})$ of \mathbf{S} given by diffeomorphisms of \mathbf{S} that preserve (not necessarily pointwise) the boundary, the set of marked points \mathbf{M} and the set of punctures \mathbf{P} . Denote by

$$\text{MCG}_{\times}(\mathbf{S}) \cong \text{MCG}_{\bullet}(\mathbf{S}) \ltimes (\mathbb{Z}_2)^p$$

the tagged mapping class group of \mathbf{S} consisting of elements (φ, δ) , where $\varphi \in \text{MCG}_{\bullet}(\mathbf{S})$ and δ is a $\{\pm 1\}$ -sign on each point in \mathbf{P} . The action of this group on the set of tagged arcs induces an embedding of $\text{MCG}_{\times}(\mathbf{S})$ into the group of cluster algebra automorphisms. It is not clear that each cluster algebra automorphism is induced by an element from $\text{MCG}_{\times}(\mathbf{S})$.

Instead of studying cluster algebra automorphisms, we change the point of view in this paper and propose to investigate automorphisms of the corresponding cluster category $\mathcal{C}(\mathbf{S})$. Amiot constructed in [1] the 2-Calabi-Yau cluster category for each (Jacobi-finite) cluster algebra \mathcal{A} ; Plamondon [14] shows that the cluster-tilting objects correspond bijectively to the clusters in \mathcal{A} . For the case of a marked surface, the set

$\mathbf{A}^\times(\mathbf{S})$ of (simple) tagged arcs correspond bijectively to the set $\mathcal{C}^\times(\mathbf{S})$ of (rigid) objects forming the cluster-tilting objects, and we denote by

$$\mathrm{Aut}_0 \mathcal{C}(\mathbf{S}) = \mathrm{Aut} \mathcal{C}(\mathbf{S}) / \mathrm{stab} \mathcal{C}^\times(\mathbf{S})$$

the group of auto-equivalences of $\mathcal{C}(\mathbf{S})$ modulo the subgroup of automorphisms that preserve the set $\mathcal{C}^\times(\mathbf{S})$. This group is isomorphic to the group of cluster algebra automorphisms studied in [2]. Any element in $\mathrm{MCG}_\times(\mathbf{S})$ acts on the set $\mathbf{A}^\times(\mathbf{S})$ and hence on $\mathcal{C}^\times(\mathbf{S})$, which induces an embedding

$$\mathrm{MCG}_\times(\mathbf{S}) \hookrightarrow \mathrm{Aut}_0 \mathcal{C}(\mathbf{S}).$$

The category $\mathcal{C}(\mathbf{S})$ admits a distinguished automorphism, the Auslander Reiten translation, which in this context coincides with the suspension functor of the triangulated category $\mathcal{C}(\mathbf{S})$. The first aim of this article is to show that the Auslander-Reiten translation can be realized by an element in the tagged mapping class group $\mathrm{MCG}_\times(\mathbf{S})$. For each boundary component Y with m marked points, denote by ρ_Y the m -th root of the Dehn twist around Y , that is, simultaneous rotation to the next marked point on Y . Then we define the (universal) tagged rotation ϱ as the permutation on $\mathbf{A}^\times(\mathbf{S})$ induced by the element

$$\varrho = \prod_{Y \subset \partial \mathbf{S}} \rho_Y \cdot \prod_{P \in \mathbf{P}} \delta^P$$

in $\mathrm{MCG}_\times(\mathbf{S})$ where the first product is over all connected components Y of $\partial \mathbf{S}$ and the second product is a simultaneous change of tagging. We show (Theorem 3.8) that the tagged rotation $\varrho \in \mathrm{MCG}_\times(\mathbf{S})$ on $\mathbf{A}^\times(\mathbf{S})$ becomes the shift [1] on $\mathcal{C}^\times(\mathbf{S})$.

This result is known from [4] for the case where there are no punctures. In fact, the cluster category of an unpunctured surface is explicitly given by a combinatorial description of string and band objects, and the shift functor [1] is well-known from the theory of string algebras. These methods are not available in the general situation since there is no explicit description of the (indecomposable) objects and morphisms yet. Instead, we show that one can always choose a triangulation such that the operation of the shift functor in the category $\mathcal{C}(\mathbf{S})$ corresponds to the tagged flip of an arc. The proof is thus an interplay between the triangulated structure of the category $\mathcal{C}(\mathbf{S})$ and the structure of triangles on the surface \mathbf{S} . In fact, the result we obtained helps to understand the objects and morphisms of $\mathcal{C}(\mathbf{S})$ and a further study aiming to generalize results in [4] to the unpunctured case is undertaken in [17].

One motivation to study such a geometric realization of the shift functor comes from physics: to compute the complete spectrum of a BPS particle, one studies maximal green sequences which go from a triangulation to its shift in $\mathcal{C}(\mathbf{S})$, see [6]. Our result allows to determine the tagged triangulation for the endpoint of any maximal green mutation sequence from the tagged triangulation of the starting point, see [5] (cf. [16]).

An invariant of the tagged rotation is the *order* for a tagged arc α , that is, the minimal natural number satisfying $X_\alpha[m] = X_\alpha$, where X_α is the object corresponding to α in $\mathcal{C}(\mathbf{S})$. By convention, it is infinite if such an m does not exist. This allows us to study the Seidel-Thomas braid group $\mathrm{Br} \mathbf{S}$ generated by spherical twists, which is the subgroup of the autoequivalence group of the 3-Calabi-Yau derived category $\mathcal{D}(\Gamma_{\mathbf{S}})$ for the quivers with potential associated to the surface. We show (Theorem 4.4) that

the intersection of the shifts and $\text{Br } \mathbf{S}$ is empty, unless \mathbf{S} is a polygon with at most one puncture (i.e. of type A or D).

Having realized one distinguished automorphism of $\mathcal{C}(\mathbf{S})$ geometrically, we will show in the sequel [3] that any cluster automorphism is induced from elements in the corresponding tagged mapping class group except for two cases (namely type D_4 and \widetilde{D}_4).

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2. PRELIMINARIES

2.1. Quiver with potential from surface. Throughout the article, \mathbf{S} denotes a *marked surface* in the sense of [9], that is, a connected Riemann surface with a fixed orientation endowed with a finite set of marked points \mathbf{M} on the boundary $\partial\mathbf{S}$ and a finite set of punctures \mathbf{P} inside \mathbf{S} such that each connected component of the boundary of \mathbf{S} contains at least one marked point. Unless otherwise stated, we will always suppose that $\partial\mathbf{S} \neq \emptyset$.

Curves in \mathbf{S} are considered up to isotopy with respect to the sets of marked points and punctures. Two curves are called *compatible* if they do not intersect in their relative interior. A *simple* curve has no self-intersection except possibly on the endpoints. An *arc* in \mathbf{S} is a simple curve whose both endpoints are marked points or punctures and which is not isotopic to a boundary component.

An *ideal triangulation* \mathbf{T} of \mathbf{S} is a maximal collection of pairwise compatible arcs in \mathbf{S} . Any triangulation \mathbf{T} of \mathbf{S} consists of

$$n = 6g + 3p + 3b + m - 6 \quad (2.1)$$

arcs (ordinary or tagged), where g is the genus of \mathbf{S} , b is the number of boundary components and m and p denote the number of marked points and punctures. The number n is called the *rank* of the surface \mathbf{S} . To exclude a few cases where the surface does not admit a triangulation we assume from now on that $n \geq 1$.

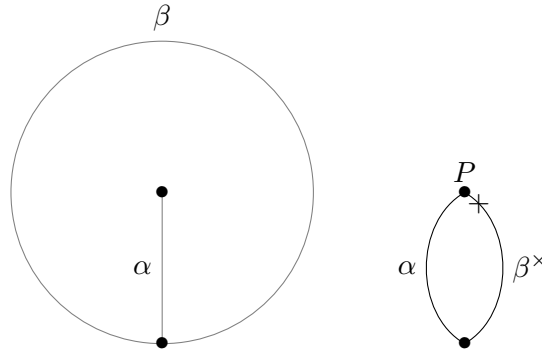


FIGURE 1. The self-folded triangle and the corresponding tagged version

The *flip* of an arc α in a triangulation \mathbf{T} is the unique arc $\alpha' \neq \alpha$ which forms a triangulation with the remaining arcs of \mathbf{T} . We denote by $\text{EG}^\circ(\mathbf{S})$ the *exchange graph*

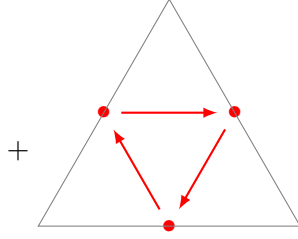


FIGURE 2. The quiver associated to a triangle

of ideal triangulations which is formed by the ideal triangulations of \mathbf{S} , with an edge between \mathbf{T} and \mathbf{T}' whenever \mathbf{T}' is obtained from \mathbf{T} by the flip of an arc. An arc in an ideal triangulation can always be flipped except when it is the arc α connecting to the internal puncture of a self-folded triangle as in the left side of figure 1. To overcome this shortcoming, tagging is introduced in [9]. So there are tagged arcs, tagged triangulations and tagged flips. The corresponding exchange graph of tagged triangulations is denoted by $\text{EG}^\times(\mathbf{S})$. Note that $\text{EG}^\times(\mathbf{S})$ can be obtained by gluing 2^p copies of $\text{EG}^\circ(\mathbf{S})$ (cf. [9]).

Definition 2.1. We have the following data associated to a tagged triangulation \mathbf{T}_\times :

- The signed adjacency matrix $B = B(\mathbf{T}_\times)$ ([9]). The rows and columns of the matrix are naturally labeled by the arcs in \mathbf{T}_\times (say from 1 to n). It is skew-symmetric and all its entries b_{ij} are in $\{0, \pm 1, \pm 2\}$. See [9, Definition 4.1] for the details.
- The quiver $Q = Q(\mathbf{T}_\times)$ corresponding to $B(\mathbf{T}_\times)$ is the quiver whose vertices are the arcs in \mathbf{T}_\times and the number of arrows from i to j equals b_{ij} ([9]). For instance, the quiver for a triangle is shown in Figure 2.
- The potential $W = W(\mathbf{T}_\times)$ is a sum of certain cycles in the complete path algebra $\widehat{\mathbf{k}Q}$, where \mathbf{k} is an algebraic closed field ([7]). Note that this potential is rigid (and thus non-degenerated) if $\partial\mathbf{S} \neq \emptyset$.

2.2. Jacobian algebra and Ginzburg algebra. Let Q be a finite quiver and W a potential on Q . The *Jacobian algebra* of the quiver with potential (Q, W) , denoted by $J(Q, W)$, is the quotient of the complete path algebra $\widehat{\mathbf{k}Q}$ by the closure of the ideal generated by $\partial_a W$, where a runs over all arrows of Q . The pair (Q, W) is called *Jacobi-finite* provided $J(Q, W)$ is finite-dimensional. The quiver with potential associated to a surface with non-empty boundary is Jacobi-finite by [7]. The Jacobian algebra can be seen as the zeroth cohomology of the *Ginzburg dg algebra* $\Gamma = \Gamma(Q, W)$ of (Q, W) which is constructed as follows ([11, Section 7.2]):

Let \overline{Q} be the graded quiver with the same vertices as Q and whose arrows are the arrows in Q in degree 0, an arrow $a^* : j \rightarrow i$ with degree -1 for each arrow $a : i \rightarrow j$ in Q and a loop $e^* : i \rightarrow i$ with degree -2 for each vertex e in Q .

The underlying graded algebra of $\Gamma(Q, W)$ is the completion of the graded path algebra $\mathbf{k}\overline{Q}$. The differential d of $\Gamma(Q, W)$ is the unique continuous linear endomorphism homogeneous of degree 1 which satisfies the Leibniz rule and takes the following values

on the arrows of \overline{Q} :

$$\begin{cases} d a = 0, & d a^* = \partial_a W, & \forall a \in Q_1, \\ d \sum_{e \in Q_0} e^* = \sum_{a \in Q_1} [a, a^*]. \end{cases}$$

Denote by $\mathcal{D}_{fd}(\Gamma)$ and $\text{per}(\Gamma)$ the *finite dimensional derived category* of $\Gamma(Q, W)$ and the *perfect derived category* of $\Gamma(Q, W)$, respectively (cf. [11, Section 7.3]). In case (Q, W) is Jacobi-finite, Amiot introduced in [1] the *generalized cluster category* $\mathcal{C}(\Gamma)$ as the quotient category $\text{per}(\Gamma)/\mathcal{D}_{fd}(\Gamma)$. In other words, there is an exact sequence of triangulated categories

$$0 \rightarrow \mathcal{D}_{fd}(\Gamma) \rightarrow \text{per}(\Gamma) \rightarrow \mathcal{C}(\Gamma) \rightarrow 0, \quad (2.2)$$

A *cluster tilting set* $\mathcal{P} = \{P_j\}_{j=1}^n$ in $\mathcal{C}(\Gamma)$ is an Ext-configuration, i.e. a maximal collection of non-isomorphic indecomposables such that $\text{Ext}^1(P_i, P_j) = 0$. Note that n is the number of vertices in Q . The *mutation* μ_i at the i -th object acts on a cluster tilting set $\{P_j\}_{j=1}^n$, by replacing P_i by another object P'_i , which can be calculated as follows (cf. [1])

$$P_i^\# = \text{Cone}(P_i \rightarrow \bigoplus_{j \neq i} \text{Irr}(P_i, P_j)^* \otimes P_j) \quad (2.3)$$

or

$$P_i^b = \text{Cone}(\bigoplus_{j \neq i} \text{Irr}(P_j, P_i) \otimes P_j \rightarrow P_i)[-1]. \quad (2.4)$$

where $\text{Irr}(P_i, P_j)$ is the space of irreducible maps $P_i \rightarrow P_j$ in the additive subcategory \mathbf{P} of $\mathcal{C}(\Gamma)$, for $\mathbf{P} = \bigoplus_{i=1}^n P_i$.

A key property of mutation is that it is an involution, i.e. we have

$$\mu_i^2(\mathcal{P}) = \mathcal{P}, \quad \forall i.$$

If $\Gamma = \Gamma(Q, W)$ is given by the quiver and potential associated to a triangulation \mathbf{T} of \mathbf{S} , then the cluster category $\mathcal{C}(\Gamma)$ is (up to derived equivalence) independent of the choice of the triangulation \mathbf{T} (cf. say [11]). We therefore write simply $\mathcal{C}(\mathbf{S})$ in this case. Moreover, we denote by $\text{CEG}_*(\mathbf{S})$ the *exchange graph* of clusters in $\mathcal{C}(\mathbf{S})$, that is, the (unoriented) graph whose vertices are cluster tilting sets and whose edges are the mutations. Let $\mathcal{C}^\times(\mathbf{S})$ be the set consisting of objects that appear in some cluster tilting set \mathcal{P} in $\text{CEG}_*(\mathbf{S})$.

2.3. The canonical bijection. By the correspondence between tagged arcs and cluster variables ([9, Theorem 7.11]) and the correspondence between cluster variables and reachable rigid objects ([14, Corollary 3.9]), we have the following canonical bijection.

Lemma 2.2. *There is a canonical bijection*

$$\zeta: \mathbf{A}^\times(\mathbf{S}) \rightarrow \mathcal{C}^\times(\mathbf{S})$$

which induces the isomorphism (between graphs)

$$\text{EG}^\times(\mathbf{S}) \cong \text{CEG}_*(\mathbf{S}), \quad (2.5)$$

which sends a tagged triangulation \mathbf{T}_\times to the cluster tilting set consisting of objects $\{\zeta(\alpha) \mid \alpha \in \mathbf{T}_\times\}$ and a tagged flip in $\text{EG}^\times(\mathbf{S})$ to a mutation in $\text{CEG}_(\mathbf{S})$.*

3. THE TAGGED ROTATION

3.1. Tagged mapping class group. Denote by $\text{MCG}_\bullet(\mathbf{S})$ the *marked mapping class group* of \mathbf{S} consisting of orientation-preserving diffeomorphisms of \mathbf{S} , up to isotopy, that preserve (*not necessary pointwise*) the boundary $\partial\mathbf{S}$, the set of marked points \mathbf{M} and the set of punctures \mathbf{P} . Denote by $\text{MCG}_\times(\mathbf{S})$ the tagged mapping class group of \mathbf{S} consisting of elements (φ, δ) , where $\varphi \in \text{MCG}_\bullet(\mathbf{S})$ and δ is a $\{\pm 1\}$ -sign on each point in \mathbf{P} . As in [2, Lemma 4.6], we have

$$\text{MCG}_\times(\mathbf{S}) \cong \text{MCG}_\bullet(\mathbf{S}) \ltimes (\mathbb{Z}_2)^p$$

where the multiplication is defined by

$$(\varphi', \delta') \circ (\varphi, \delta) = (\varphi' \circ \varphi, (\delta' \circ \varphi) \cdot \delta)$$

using the action

$$((\delta' \circ \varphi) \cdot \delta)(P) = \delta'(\varphi(P)) \cdot \delta(P).$$

We can identify $\text{MCG}_\bullet(\mathbf{S})$ with

$$\text{MCG}_\bullet(\mathbf{S}) \times \mathbf{1} \subset \text{MCG}_\times(\mathbf{S}),$$

where $\mathbf{1}(P) = 1$ for any $P \in \mathbf{P}$.

Denote by $\text{Aut } \mathcal{C}(\mathbf{S})$ the group of auto-equivalences of $\mathcal{C}(\mathbf{S})$ and by $\text{stab } \mathcal{C}^\times(\mathbf{S})$ the subgroup of $\text{Aut } \mathcal{C}(\mathbf{S})$ that preserves the set $\mathcal{C}^\times(\mathbf{S})$. Let $\text{Aut}_0 \mathcal{C}(\mathbf{S}) = \text{Aut } \mathcal{C}(\mathbf{S}) / \text{stab } \mathcal{C}^\times(\mathbf{S})$.

Any element in $\text{MCG}_\times(\mathbf{S})$ acts on the set $\mathbf{A}^\times(\mathbf{S})$ and hence on $\mathcal{C}^\times(\mathbf{S})$ via the canonical bijection ζ in Lemma 2.2. The following lemma, proved in [2] for cluster algebras, is an analogue in terms of the cluster category.

Lemma 3.1. *There is a canonical injection $\iota_{\mathbf{S}}: \text{MCG}_\times(\mathbf{S}) \hookrightarrow \text{Aut}_0 \mathcal{C}(\mathbf{S})$.*

The fixed orientation \mathcal{O} of \mathbf{S} induces an orientation \mathcal{O} on each component of $\partial\mathbf{S}$. This allows to define a rotation ρ on the set of marked points \mathbf{M} in $\partial\mathbf{S}$ which sends each marked point M to the next marked point M' in $\partial\mathbf{S}$ along \mathcal{O} (cf. Figure 3). We would like to extend this rotation to an element in $\text{MCG}_\bullet(\mathbf{S})$:

Example 3.2. For each component Y in the boundary $\partial\mathbf{S}$ with $m_Y = |\mathbf{M} \cap Y|$ marked points there is a rotation ρ_Y in $\text{MCG}_\bullet(\mathbf{S}) \subset \text{MCG}_\times(\mathbf{S})$ which sends each marked point

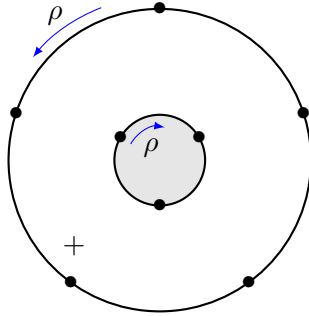


FIGURE 3. The rotation on \mathbf{S}

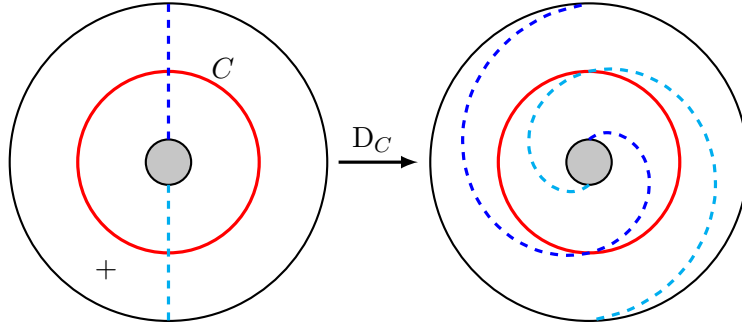


FIGURE 4. The Dehn twist

M in Y to the next marked point in Y . Note that $\rho_Y^{m_Y}$ is the (positive) Dehn twist D_Y along Y (cf. Figure 4).

Further, ρ_Y induces a permutation on $\mathbf{A}^\times(\mathbf{S})$ (cf. the left picture in Figure 6). For any tagged arc $\alpha \in \mathbf{A}^\times(\mathbf{S})$ with the normal arc \widehat{AC} and end points A and C , the normal arc of $\rho_Y(\alpha)$ is the arc connecting $B = \rho_Y(A)$ and $D = \rho_Y(C)$ that can contract to the union

$$\widehat{BA} \cup \widehat{AC} \cup \widehat{CD},$$

\widehat{BA} and \widehat{CD} are the boundary arc segments in Y . Moreover, the tagging of $\rho_Y(\alpha)$ at any puncture inherits the tagging of α at that puncture.

Example 3.3. for each puncture $P \in \mathbf{P}$, there is a *tagging switch* $\delta^P = (\text{id}, \delta_P)$ in $\text{MCG}_\times(\mathbf{S})$, such that $\delta_P(P') = -1$ if and only if $P' = P$.

Further, δ^P induces a permutation on $\mathbf{A}^\times(\mathbf{S})$ that preserves the underlying arcs but changes the tagging at the puncture P .

Definition 3.4. The (*universal*) *tagged rotation* ϱ is the permutation on $\mathbf{A}^\times(\mathbf{S})$ induced by the element

$$\varrho = \prod_{Y \subset \partial \mathbf{S}} \rho_Y \cdot \prod_{P \in \mathbf{P}} \delta^P \quad (3.1)$$

in $\text{MCG}_\times(\mathbf{S})$ where the first product is over all connected components Y of $\partial \mathbf{S}$. Note that the order in (3.1) does not matter since the ρ_Y commute.

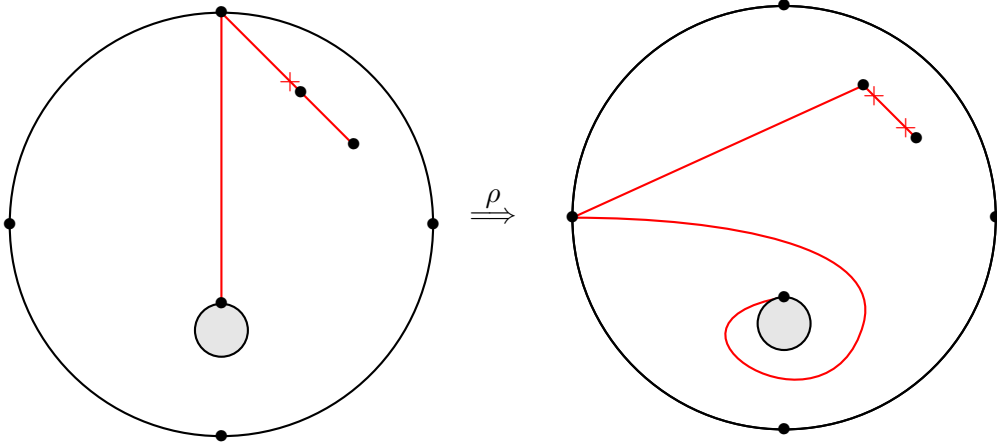
An instance of a tagged rotation is shown in Figure 5.

3.2. AR-translation on marked surfaces. This subsection is devoted to show that the tagged rotation corresponds to the shift in the cluster category via the bijection ζ in Lemma 2.2.

Lemma 3.5. *For a tagged arc $\alpha \in \mathbf{A}^\times(\mathbf{S})$ whose end points are distinct and at least one of them is not a puncture, we have $\zeta_{\varrho(\alpha)} = \zeta_\alpha[1]$.*

Proof. By formula (2.3), for any cluster tilting set $\mathcal{P} = \{P_j\}_{j=1}^n$, an object P_i in \mathcal{P} becomes $P_i[1]$ in $\mu_i(\mathcal{P})$ if

$$\text{Irr}(P_i, P_j) = 0, \quad \forall j. \quad (3.2)$$

FIGURE 5. The tagged rotation on arcs in \mathbf{S}

Since irreducible morphisms between the objects P_k 's correspond to the arrows of Gabriel quiver $Q_{\mathcal{P}}$ of

$$\text{End}\left(\bigoplus_{k=1}^n P_k, \bigoplus_{k=1}^n P_k\right),$$

condition (3.2) is equivalent to P_i being a source in $Q_{\mathcal{P}}$. Thus, via the correspondence ζ in Lemma 2.2, we only need to show that there exists a tagged triangulation \mathbf{T}_{\times} containing α such that α is a sink in the corresponding associated quiver and $\varrho(\alpha)$ is the tagged arc obtained by the tagged flip of \mathbf{T}_{\times} at α .

Without loss of generality, suppose that α is not tagged at either end. We have two cases.

- I. If $\alpha = \widehat{AC}$ such that $A, C \in \mathbf{M}$, we denote $B = \varrho(A)$ and $D = \varrho(C)$. Choose a triangulation \mathbf{T}_{\times} containing the triangles $\triangle ABC$ and $\triangle ACD$ as shown in the left picture of Figure 6.
- II. If $\alpha = \widehat{AP}$ such that $A \in \mathbf{M}$ and $P \in \mathbf{P}$, we denote $B = \varrho(A)$. Choose a triangulation \mathbf{T}_{\times} containing α and the arc \widehat{AB} that only encloses the puncture P . Moreover, \mathbf{T}_{\times} can either contain the normal arc \widehat{BP} or the tagged arc \widehat{AP} , as shown in the middle and right pictures of Figure 6.

In both cases it is straightforward to see that α corresponds to a source in the quiver $Q_{\mathcal{P}}$ and the tagged rotation $\varrho(\alpha)$ of α is indeed the tagged arc obtained by the tagged flip of \mathbf{T}_{\times} at α , as required. \square

Let α_i be the arc labelled with i in the upper left picture in Figure 7. The blue triangle is the rest of the surface, i.e. might contain punctures, boundary components and handle bodies. For instance, it is possible that $\alpha_4 = \alpha_5$ and \mathbf{S} is just a torus with one boundary component and one marked point. We consider certain arcs in a higher genus surface, namely a non-separating loop with endpoint in \mathbf{M} , such as α_1 . Note

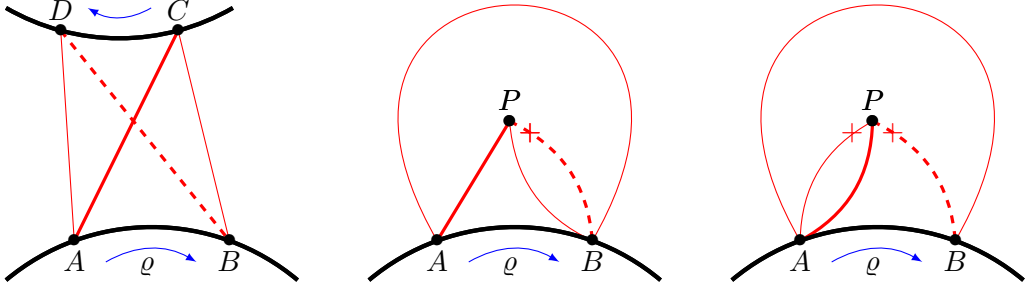


FIGURE 6. The flip/tagged rotation of certain types of arcs

that α_1 is non-trivial in the homology group even if we glue a disc to each boundary component and ignore all the marked points.

Lemma 3.6. *For $\alpha = \alpha_1$ as shown in the upper left picture in Figure 7, we have $\zeta_{\varrho(\alpha)} = \zeta_\alpha[1]$.*

Proof. Let X_i be the corresponding rigid object ζ_{α_i} in the cluster category. We have a mutation sequence $\mu_{321} = \mu_3 \circ \mu_2 \circ \mu_1$ as shown in Figure 7, starting from the upper left picture to the lower left one, then the middle one and finally the right one. The mutation sequence of the corresponding quivers is shown in Figure 8. The corresponding exchange relations of those three mutations are

$$\begin{aligned} X_1 &\rightarrow X_2 \rightarrow X_7 \rightarrow X_1[1], \\ X_2 &\rightarrow X_3 \oplus X_7 \rightarrow X_8 \rightarrow X_2[1], \\ X_3 &\rightarrow X_8 \rightarrow X_9 \rightarrow X_3[1]. \end{aligned}$$

The first two triangles fit into the following diagram of triangles

$$\begin{array}{ccccccc} & & X_2 & \xlongequal{\quad} & X_2 & & \\ & & \downarrow & & \downarrow & & \\ X_3 & \longrightarrow & X_3 \oplus X_7 & \longrightarrow & X_7 & \longrightarrow & X_3[1] \\ & & \downarrow & & \downarrow & & \\ \parallel & & & & & & \parallel \\ X_3 & \longrightarrow & X_8 & \cdots\cdots\cdots & X_1[1] & \cdots\cdots\cdots & X_3[1] \\ & & \downarrow & & \downarrow & & \\ & & X_2[1] & \xlongequal{\quad} & X_2[1] & & \end{array} \quad (3.3)$$

It is straightforward to see that the leftmost and the top square are commutative. For instance, the map from X_2 to X_7 in the top square is the irreducible map corresponding to the arrow $7 \rightarrow 2$ in the second quiver of Figure 8. By the Octahedral Axiom, the

third line (with dotted arrows) in (3.3) is a triangle, which coincides with the third exchange relation. In other words, $X_9 = X_1[1]$. \square

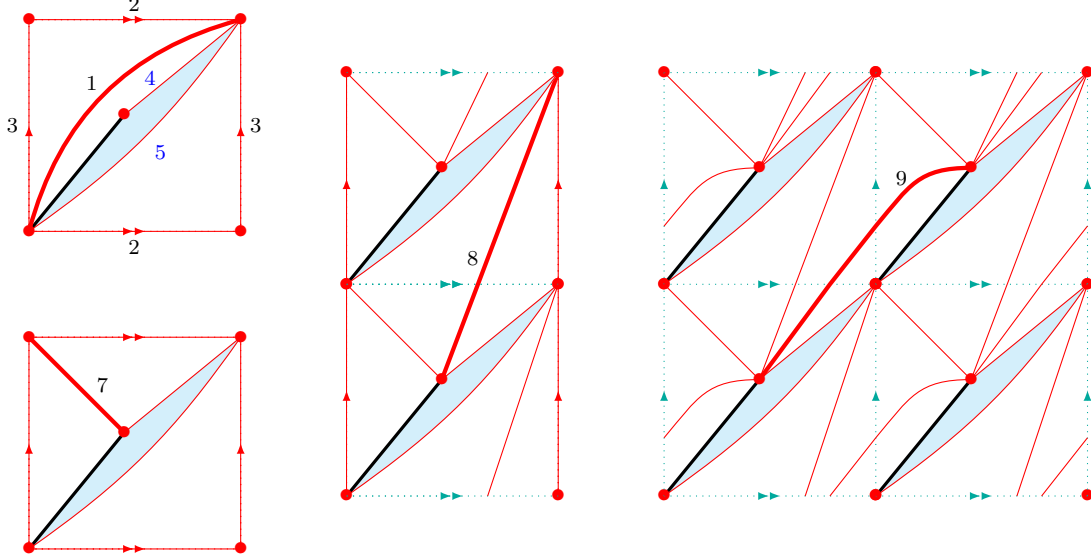


FIGURE 7. The mutation sequence μ_{321} on a higher genus surface

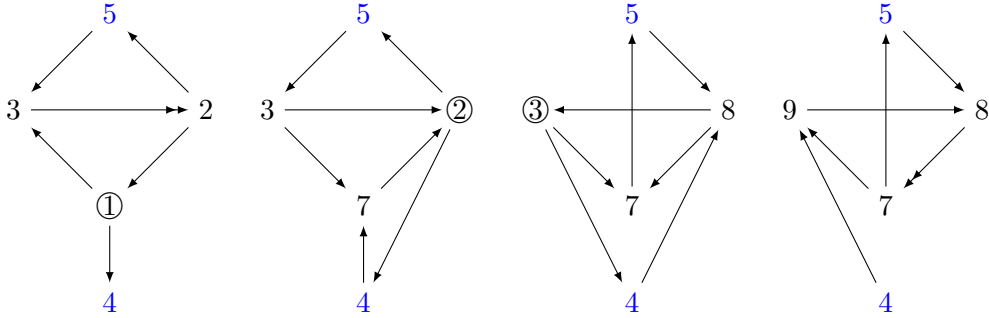
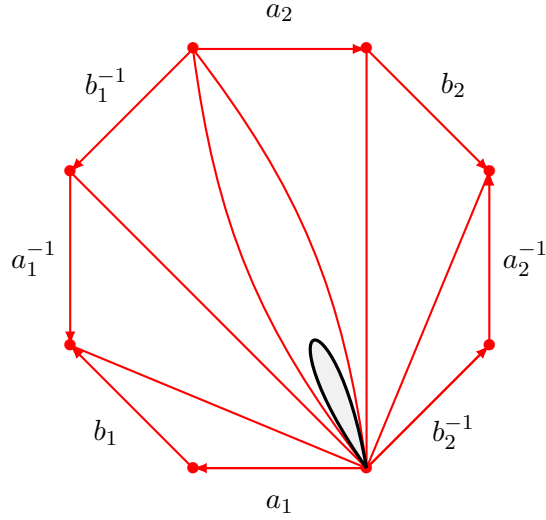
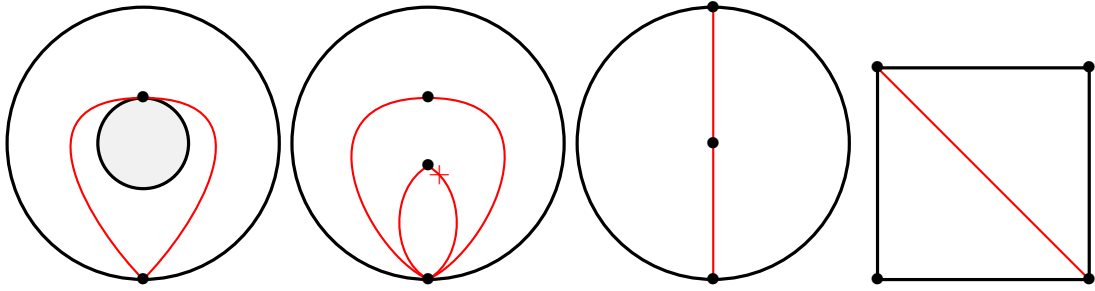


FIGURE 8. The mutation sequence μ_{321} of the quiver

Remark 3.7. Presumably, it is possible to use the same method as in Lemma 3.6 for all cases of tagged arcs. However, it is complicated when the arc connecting two punctures. Instead, we use the tagged mapping class group to give a simple proof.

Theorem 3.8. *The tagged rotation $\varrho \in \text{MCG}_\times(\mathbf{S})$ on $\mathbf{A}^\times(\mathbf{S})$ becomes the shift $[1]$ on $\mathcal{C}^\times(\mathbf{S})$, i.e. $\iota_{\mathbf{S}}(\varrho) = [1]$.*

Proof. We only need to show that there exists a tagged triangulation \mathbf{T}_\times consisting of the types of arcs considered in Lemma 3.5 and Lemma 3.6. If so, since the action on

FIGURE 9. The basic case **I**: $4g$ -gon presentation of a genus $g = 2$ surfaceFIGURE 10. The basic cases **II**, **III**, **IV**, **V**

$\mathcal{C}^\times(\mathbf{S})$ of an element in $\text{Aut}_0 \mathcal{C}(\mathbf{S})$ is determined by the action on a particular cluster tilting set, the theorem follows by Lemma 3.1.

Any marked surface belongs to at least one of the following categories:

- I.** $g > 0$,
- II.** $g = 0$, $b \geq 2$,
- III.** $g = 0$, $b = 1$, $p \geq 2$,
- IV.** $g = 0$, $b = 1$, $p = 1$, $m \geq 2$,
- V.** $g = 0$, $b = 1$, $p = 0$ and $m \geq 4$.

Use induction, starting from one of the following basic cases

- i.** $g > 0$, $b = m = 1$ and $p = 0$,
- ii.** $g = 0$, $b = m = 2$ and $p = 0$,
- iii.** $g = 0$, $b = m = 1$ and $p = 2$,
- iv.** $g = 0$, $b = p = 1$ and $m = 2$,
- v.** $g = 0$, $b = 1$, $p = 0$ and $m = 4$,

any marked surface can be obtained from one of the corresponding basic cases by adding

- a marked point on an existing boundary component;
- a boundary component with one marked point;
- or a puncture.

Figure 9 shows that there exists a tagged triangulation for the basic case **i**, consisting of the two types of arc in Lemma 3.5 and Lemma 3.6. The same holds for the basic cases **ii**, **iii**, **iv** and **v** as shown in Figure 10. Notice that in each of these basic cases, there is a triangle with a boundary arc. Figure 11 shows that in the each of the three cases of adding, we can modify the tagged triangulation \mathbf{T}_\times of the old surface (within the triangle with a boundary arc) to get a tagged triangulation \mathbf{T}'_\times for the new surface, such that \mathbf{T}'_\times only consists of those types of arcs. Thus we are done.

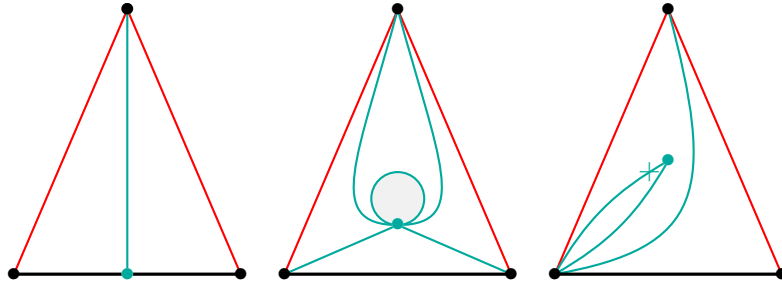


FIGURE 11. Three cases of adding (in green)

□

4. APPLICATIONS

4.1. On the Jacobian algebras.

Lemma 4.1. *[Amiot, Keller-Reiten] There is a canonical bijection*

$$\theta_{\mathbf{T}}: \mathcal{C}(\mathbf{S}) / \text{add } \zeta(\mathbf{T}) \rightarrow J(\mathbf{T}).$$

By the proof in [12, Section 3.5], the AR-translation on $J(\mathbf{T})$ is induced from the AR-translation (or shift [1]) of the cluster category $\mathcal{C}(\mathbf{S})$. Thus we have the following corollary.

Corollary 4.2. *Under the bijection $\theta_{\mathbf{T}}$ in Lemma 4.1, the tagged rotation ϱ on $\mathbf{A}^\times(\mathbf{S})$ induces the AR-translation τ on the objects in $J(\mathbf{T})$ that are the images of the reachable rigid objects.*

4.2. Shifts and Seidel-Thomas braid group. Let $\Gamma = \Gamma(Q, W)$ be the Ginzburg dg algebra of any quiver with potential. It is known that $\mathcal{D}_{fd}(\Gamma)$ is a Calabi-Yau-3 category, which admits a standard heart \mathcal{H}_Γ generated by simple Γ -modules S_e , for $e \in Q_0$, each of which is a 3-spherical objects. Recall that an object S is N -spherical when $\text{Hom}^\bullet(S, S) = \mathbf{k} \oplus \mathbf{k}[-N]$. Moreover, we recall (e.g. [18]) a distinguished family of auto-equivalences of $\mathcal{D}_{fd}(\Gamma)$.

Definition 4.3. The *twist functor* ϕ of a spherical object S is defined by

$$\phi_S(X) = \text{Cone}(X \rightarrow S \otimes \text{Hom}^\bullet(X, S)^\vee)[-1] \quad (4.1)$$

with inverse

$$\phi_S^{-1}(X) = \text{Cone}(S \otimes \text{Hom}^\bullet(S, X) \rightarrow X) \quad (4.2)$$

Note that the graded dual of a graded \mathbf{k} -vector space $V = \bigoplus_{j \in \mathbb{Z}} V_j[j]$ is $V^\vee = \bigoplus_{j \in \mathbb{Z}} V_j^*[-i]$. The *Seidel-Thomas braid group*, denoted by $\text{Br } \Gamma$, is the subgroup of $\text{Aut } \mathcal{D}_{fd}(\Gamma)$ generated by the twist functors of the simples in $\text{Sim } \mathcal{H}_\Gamma$.

We further collect some terminology from [13]. Denote by $\text{EG}^\circ(\mathcal{D}(\Gamma_{\mathbf{S}}))$ the principal component of the exchange graph of hearts in $\mathcal{D}(\Gamma_{\mathbf{S}})$, which consists of all hearts that can be obtained through iterated (simple) tilts from the standard heart \mathcal{H}_Γ . Denote by $\text{CEG}(\mathbf{S})$, the *oriented exchange graph* of $\mathcal{C}(\mathbf{S})$, which can be obtained from $\text{CEG}_*(\mathbf{S})$ by replacing each unoriented edge with a 2-cycle. By [1] (cf. [13, Theorem 8.6]), there is a map

$$v : \text{EG}^\circ(\mathcal{D}(\Gamma_{\mathbf{S}})) \rightarrow \text{CEG}(\mathbf{S}) \quad (4.3)$$

which induces (cf. [10] for the general case and [13] for the acyclic case) an isomorphism

$$\text{CEG}(\mathbf{S}) = \text{EG}^\circ(\mathcal{D}(\Gamma_{\mathbf{S}})) / \text{Br}.$$

More precisely, the map v goes as follows. Let \mathcal{H} be a heart in $\text{EG}^\circ(\mathcal{D}(\Gamma_{\mathbf{S}}))$ which corresponds to a t-structure \mathcal{P} in $\mathcal{D}_{fd}(\Gamma)$. Lift \mathcal{P} to a t-structure in $\text{per}(\Gamma)$ via the inclusion in (2.2) which corresponds to a silting object. The image of the silting object is defined to be $v(\mathcal{H})$. Furthermore, it is clear that v commutes with the shift functor, i.e.

$$v(\mathcal{H}[1]) = v(\mathcal{H})[1].$$

Theorem 4.4. *Let (Q, W) be a quiver with potential associated to some triangulation of a marked surface \mathbf{S} and $\Gamma_{\mathbf{S}}$ be the corresponding Ginzburg dg algebra. If \mathbf{S} is not a polygon or a once-punctured polygon, i.e. (Q, W) is not mutation-equivalent to a quiver of type A or D, then the shifts $\mathbb{Z}[1]$ do not intersect $\text{Br } \Gamma_{\mathbf{S}}$.*

Proof. Suppose that $[m] \in \text{Br } \Gamma_{\mathbf{S}}$ for some integer m . Since the induced action of the braid group on $\text{CEG}(\mathbf{S})$ is trivial, $[m]$ acts trivially on $\text{CEG}(\mathbf{S})$, or $[m] = \text{id}$ in $\text{Aut}_0 \mathcal{C}(\mathbf{S})$. On the other hand, the tagged rotation of any arc connecting different boundary components or dividing the set of punctures \mathbf{P} into two parts, have infinite order. Thus \mathbf{S} can have at most one boundary component and one puncture. Similar, the genus of \mathbf{S} must be zero since that the Dehn twist of a boundary component in a higher genus surface has infinite order. \square

4.3. Center of the braid group. In this subsection, we discuss the centers of the braid groups when Q is of type A and D.

Recall that the quasi-center of Br_Q is the subgroup of elements $\Delta(Q)$ in Br_Q satisfying $\Delta(Q) \cdot \mathbf{b} \cdot \Delta(Q)^{-1} = \mathbf{b}$, where \mathbf{b} is the standard generating set of Br_Q , and that this subgroup is an infinite cyclic group generated by a special element \tilde{z} of Br_Q , called *fundamental element*. The center $Z(\text{Br}_Q)$ of Br_Q is an infinite cyclic group. The $z_Q =$

$\Delta(Q)$ generates $Z(\text{Br}_Q)$ if Q is of type D_n for even n and $z_Q = \Delta(Q)^2$ generates $Z(\text{Br}_Q)$ if Q is of type D_n for odd n or type A. Note that there is a quotient map

$$\pi : \text{Br}_Q \rightarrow \text{Br } \Gamma_{\mathbf{S}}$$

since the spherical twists satisfy the braid relation cf. [13, (7.4)].

Example 4.5. [15] Let \mathbf{S} be an $(n+3)$ -gon as shown in the left picture of Figure 12. Then the tagged rotation has order $n+3$. Further, for the shift $[1]$ in $\mathcal{D}(\Gamma_{\mathbf{S}})$, we have

$$\pi(\Delta(Q)^2) = [n+3].$$

where Q is the quiver of type A_n .

Example 4.6. [15] Similarly, if \mathbf{S} is an n -gon with a puncture as shown in the right picture of Figure 12, then the tagged rotation has order n if n is even and order $2n$ if n is odd. Further, for the shift $[1]$ in $\mathcal{D}(\Gamma_{\mathbf{S}})$, we have

$$\pi(\Delta(Q)) = [n],$$

where Q is the quiver of type D_n .

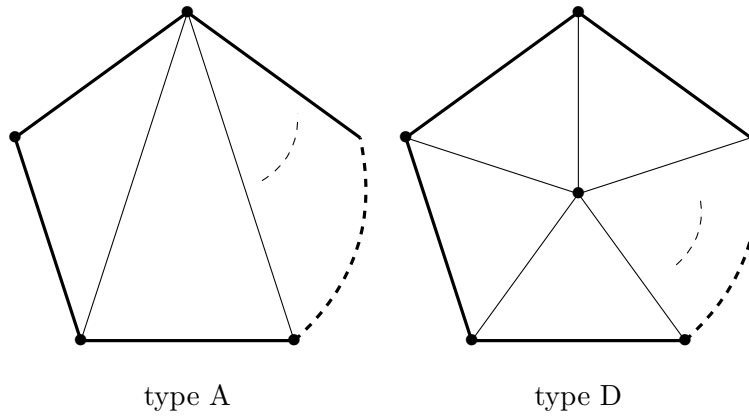


FIGURE 12. The triangulations

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